Maximum Likelihood Estimation

Goal: Estimate a population parameter \( \theta \).

Data: \( X_1, X_2, \ldots, X_n \sim \text{iid with distribution depending on } \theta \).

If one has many estimators to choose from, pick

- That with the smallest SE, among all unbiased estimators.
- That with the smallest RMS error, even if biased.

\[ \rightarrow \] Sometimes it is not clear how to form even one good estimator.
Maximum likelihood estimation

Likelihood function: \( L(\theta) = \Pr(\text{data} \mid \theta) \)

Log likelihood: \( l(\theta) = \log \Pr(\text{data} \mid \theta) \)

Maximum likelihood estimate:

\[ \to \quad \text{Choose, as the estimate of } \theta, \text{ the value of } \theta \text{ for which the likelihood function } L(\theta) \text{ (or equivalently, the log likelihood function) is maximized.} \]

You need to solve these equations analytically or numerically!

Example 1

Suppose \( X \sim \text{Binomial}(n, p) \).

Log likelihood function: \[ l(p) = \log \left\{ \binom{n}{x} p^x (1 - p)^{(n-x)} \right\} \]

\[ = x \log(p) + (n - x) \log(1 - p) + \text{constant} \]

MLE: the obvious thing: \( \hat{p} = \frac{x}{n} \)

\[ n=100, \ x=22 \]

MLE = 0.22
Example 2

Suppose $X_1, \ldots, X_{20} \sim \text{iid Poisson}(\lambda)$.

Log likelihood function:

$$l(\lambda) = \log \left\{ \prod_i e^{-\lambda} \frac{\lambda^{x_i}}{x_i!} \right\}$$

$$= \ldots = -20\lambda + (\sum x_i) \log \lambda + \text{constant}$$

MLE: the obvious thing: $\hat{\lambda} = \bar{x}$

Example 3

Suppose $X_1, \ldots, X_n \sim \text{iid N}(\mu, \sigma)$

Log likelihood function:

$$l(\mu, \sigma) = \log \left\{ \prod_i \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{x_i - \mu}{\sigma} \right)^2 \right] \right\}$$

MLEs: almost the obvious things:

$$\hat{\mu} = \bar{x} \quad \hat{\sigma} = \sqrt{\sum (x_i - \bar{x})^2 / n}$$
About MLEs

Maximum likelihood estimation is a general procedure for finding a reasonable estimator

- In many cases, the MLE turns out to be the obvious thing.

- MLEs are often very good (but not necessarily the best) possible estimators:
  - Unbiased or nearly unbiased.
  - Small standard errors.

- Sometimes obtaining the MLEs requires hefty computation!

Example 4: ABO blood groups

<table>
<thead>
<tr>
<th>Phenotype</th>
<th>Genotype</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>O</td>
<td>OO</td>
<td>$p_O^2$</td>
</tr>
<tr>
<td>A</td>
<td>AA or AO</td>
<td>$p_A^2 + 2p_A p_O$</td>
</tr>
<tr>
<td>B</td>
<td>BB or BO</td>
<td>$p_B^2 + 2p_B p_O$</td>
</tr>
<tr>
<td>AB</td>
<td>AB</td>
<td>$2p_A p_B$</td>
</tr>
</tbody>
</table>

Frequencies under the assumption of Hardy-Weinberg equilibrium.
Example 4: Data

<table>
<thead>
<tr>
<th>Phenotype</th>
<th>No. subjects</th>
<th>% subjects</th>
</tr>
</thead>
<tbody>
<tr>
<td>O</td>
<td>117</td>
<td>46.8%</td>
</tr>
<tr>
<td>A</td>
<td>98</td>
<td>39.2%</td>
</tr>
<tr>
<td>B</td>
<td>29</td>
<td>11.6%</td>
</tr>
<tr>
<td>AB</td>
<td>6</td>
<td>2.4%</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>250</strong></td>
<td><strong>100%</strong></td>
</tr>
</tbody>
</table>

What are the estimates of \(p_A\), \(p_B\), \(p_O\)?

Example 4: Estimates

Simple estimates:

\[ \hat{p}_O = \sqrt{0.468} = 0.684 \]

\[ \hat{p}_A^2 + 2\hat{p}_A 0.684 = 0.392 \quad \rightarrow \quad \hat{p}_A = 0.243 \]

\[ \hat{p}_B = 0.024 / (2\hat{p}_A) = 0.072 \]

Log likelihood

Remember the Multinomial distribution function!

\[
l(p_O, p_A, p_B) = 117 \log(p_O^2) + 98 \log(p_A^2 + 2p_A p_O) + 29 \log(p_B^2 + 2p_B p_O) + 6 \log(2p_A p_B) \]
Example 4: log likelihood

Consider the problem of estimating the recombination fraction (call that parameter $\theta$) between two genetic markers in an intercross.

Note: We won’t observe the haplotypes.
### Example 5

#### Data Probabilities

<table>
<thead>
<tr>
<th></th>
<th>AA</th>
<th>Aa</th>
<th>aa</th>
</tr>
</thead>
<tbody>
<tr>
<td>BB</td>
<td>58</td>
<td>9</td>
<td>0</td>
</tr>
<tr>
<td>Bb</td>
<td>8</td>
<td>95</td>
<td>14</td>
</tr>
<tr>
<td>bb</td>
<td>1</td>
<td>12</td>
<td>53</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>AA</th>
<th>Aa</th>
<th>aa</th>
</tr>
</thead>
<tbody>
<tr>
<td>BB</td>
<td>$\frac{1}{4} (1 - \theta)^2$</td>
<td>$\frac{1}{2} \theta(1 - \theta)$</td>
<td>$\frac{1}{4} \theta^2$</td>
</tr>
<tr>
<td>Bb</td>
<td>$\frac{1}{2} \theta(1 - \theta)$</td>
<td>$\frac{1}{2} [\theta^2 + (1 - \theta)^2]$</td>
<td>$\frac{1}{2} \theta(1 - \theta)$</td>
</tr>
<tr>
<td>bb</td>
<td>$\frac{1}{4} \theta^2$</td>
<td>$\frac{1}{2} \theta(1 - \theta)$</td>
<td>$\frac{1}{4} (1 - \theta)^2$</td>
</tr>
</tbody>
</table>

possible estimates of the recombination fraction, $\theta$?

$$L(\theta) \propto \left\{ \frac{1}{4} (1 - \theta)^2 \right\}^{(58+53)} \times \left\{ \frac{1}{2} \theta(1 - \theta) \right\}^{(9+8+14+12)} \times \left\{ \frac{1}{4} \theta^2 \right\}^{(1+0)} \times \left\{ \frac{1}{2} [\theta^2 + (1 - \theta)^2] \right\}^{95}$$

### Example 5: log likelihood function

![Graph showing log likelihood function with MLE = 9.4%](image-url)
A closer view

MLE = 9.4%