Example


Place tick on clay island surrounded by water, with two capillary tubes: one treated with deer-gland-substance; one untreated.

<table>
<thead>
<tr>
<th>Tick sex</th>
<th>Leg</th>
<th>Deer sex</th>
<th>treated</th>
<th>untreated</th>
</tr>
</thead>
<tbody>
<tr>
<td>male</td>
<td>fore</td>
<td>female</td>
<td>24</td>
<td>5</td>
</tr>
<tr>
<td>female</td>
<td>fore</td>
<td>female</td>
<td>18</td>
<td>5</td>
</tr>
<tr>
<td>male</td>
<td>fore</td>
<td>male</td>
<td>23</td>
<td>4</td>
</tr>
<tr>
<td>female</td>
<td>fore</td>
<td>male</td>
<td>20</td>
<td>4</td>
</tr>
<tr>
<td>male</td>
<td>hind</td>
<td>female</td>
<td>17</td>
<td>8</td>
</tr>
<tr>
<td>female</td>
<td>hind</td>
<td>female</td>
<td>25</td>
<td>3</td>
</tr>
<tr>
<td>male</td>
<td>hind</td>
<td>male</td>
<td>21</td>
<td>6</td>
</tr>
<tr>
<td>female</td>
<td>hind</td>
<td>male</td>
<td>25</td>
<td>2</td>
</tr>
</tbody>
</table>

→ Is the tick more likely to go to the treated tube?
Test for a proportion

Suppose $X \sim \text{Binomial}(n, p)$.

Test $H_0 : p = \frac{1}{2}$ vs $H_a : p \neq \frac{1}{2}$.

Reject $H_0$ if $X \geq H$ or $X \leq L$.

Choose $H$ and $L$ such that
\[
\Pr(X \geq H \mid p = \frac{1}{2}) \leq \frac{\alpha}{2} \quad \text{and} \quad \Pr(X \leq L \mid p = \frac{1}{2}) \leq \frac{\alpha}{2}.
\]

Thus $\Pr(\text{Reject } H_0 \mid H_0 \text{ is true}) \leq \alpha$.

The difficulty: The Binomial distribution is hard to work with. Because of its discrete nature, you can’t get exactly your desired significance level ($\alpha$).
Rejection region

Consider \( X \sim \text{Binomial}(n=29, p) \).

Test of \( H_0 : p = \frac{1}{2} \) vs \( H_a : p \neq \frac{1}{2} \) at significance level \( \alpha = 0.05 \).

Lower critical value:
\[
q\text{binom}(0.025, 29, 0.5) = 9
\]
\[
\Pr(X \leq 9) = p\text{binom}(9, 29, 0.5) = 0.031 \rightarrow L = 8
\]

Upper critical value:
\[
q\text{binom}(0.975, 29, 0.5) = 20
\]
\[
\Pr(X \geq 20) = 1 - p\text{binom}(19, 29, 0.5) = 0.031 \rightarrow H = 21
\]

Reject \( H_0 \) if \( X \leq 8 \) or \( X \geq 21 \). (For testing \( H_0 : p = \frac{1}{2}, H = n - L \))
Binomial\( (n=29, \ p=1/2) \)
Consider $X \sim \text{Binomial}(n=29, p)$.

Test of $H_0 : p = \frac{1}{2}$ vs $H_a : p \neq \frac{1}{2}$ at significance level $\alpha = 0.05$.

Reject $H_0$ if $X \leq 8$ or $X \geq 21$.

Actual significance level:

$$\alpha = \Pr(X \leq 8 \text{ or } X \geq 21 \mid p = \frac{1}{2})$$

$$= \Pr(X \leq 8 \mid p = \frac{1}{2}) + [1 - \Pr(X \leq 20 \mid p = \frac{1}{2})]$$

$$= \text{pbinom}(8, 29, 0.5) + 1 - \text{pbinom}(20, 29, 0.5)$$

$$= 0.024$$

If we used instead “Reject $H_0$ if $X \leq 9$ or $X \geq 20$”, the significance level would be

$$\text{pbinom}(9, 29, 0.5) + 1 - \text{pbinom}(19, 29, 0.5) = 0.061$$
Example 1

Observe $X = 24$ (for $n = 29$).

Reject $H_0 : p = \frac{1}{2}$ if $X \leq 8$ or $X \geq 21$.

Thus we reject $H_0$ and conclude that the ticks were more likely to go to the deer-gland-substance-treated tube.

P-value $= 2 \times \Pr(X \geq 24 \mid p = \frac{1}{2})$

$$= 2 \times (1 - \text{pbinom}(23, 29, 0.5))$$

$$= 0.0005.$$
Example 2

Observe $X = 17$ (for $n = 25$); assume $X \sim \text{Binomial}(n=25, p)$.

Test of $H_0 : p = \frac{1}{2}$ vs $H_a : p \neq \frac{1}{2}$ at significance level $\alpha = 0.05$.

Rejection rule: Reject $H_0$ if $X \leq 7$ or $X \geq 18$.

- $\text{qbinom}(0.025, 25, 0.5) = 8$
- $\text{pbinom}(8, 25, 0.5) = 0.054$
- $\text{pbinom}(7, 25, 0.5) = 0.022$

Significance level:

$$\text{pbinom}(7,25,0.5) + 1-\text{pbinom}(17,25,0.5) = 0.043$$

Since we observed $X = 17$, we fail to reject $H_0$.

$$\text{P-value} = 2 \times (1-\text{pbinom}(16,25,0.5)) = 0.11$$
Confidence interval for a proportion

Suppose \( X \sim \text{Binomial}(n=29, p) \) and we observe \( X = 24 \).

Consider the test of \( H_0 : p = p_0 \) vs \( H_a : p \neq p_0 \).

We reject \( H_0 \) if

\[
\Pr(X \leq 24 \mid p = p_0) \leq \frac{\alpha}{2} \quad \text{or} \quad \Pr(X \geq 24 \mid p = p_0) \leq \frac{\alpha}{2}
\]

95\% confidence interval for \( p \):

\( \rightarrow \) The set of \( p_0 \) for which a two-tailed test of \( H_0 : p = p_0 \) would not be rejected, for the observed data, with \( \alpha = 0.05 \).

\( \rightarrow \) The “plausible” values of \( p \).
Example 1

X ∼ Binomial(n=29, p); observe X = 24.

Lower bound of 95% confidence interval:
Largest $p_0$ such that $\Pr(X \geq 24 \mid p = p_0) \leq 0.025$

Upper bound of 95% confidence interval:
Smallest $p_0$ such that $\Pr(X \leq 24 \mid p = p_0) \leq 0.025$

$\rightarrow \text{binom.test}(24, 29)$

95% CI for $p$: (0.642, 0.942)

Note: $\hat{p} = 24/29 = 0.83$ is not the midpoint of the CI.
Example 1

Binomial(n=29, p=0.64)

Binomial(n=29, p=0.94)
Example 2

X \sim \text{Binomial}(n=25, \ p); \ observe \ X = 17.

Lower bound of 95\% confidence interval:

\[ p_L \text{ such that 17 is the 97.5 percentile of Binomial}(n=25, \ p_L) \]

Upper bound of 95\% confidence interval:

\[ p_H \text{ such that 17 is the 2.5 percentile of Binomial}(n=25, \ p_H) \]

\[ \rightarrow \ \text{binom.test}(17,25) \]

95\% CI for p: (0.465, 0.851)

Again, \( \hat{p} = 17/25 = 0.68 \) is not the midpoint of the CI
Example 2

Binomial(n=25, p=0.46)

Binomial(n=25, p=0.85)
The case $X = 0$

Suppose $X \sim \text{Binomial}(n, p)$ and we observe $X = 0$.

Lower limit of 95% confidence interval for $p$: $\rightarrow 0$

Upper limit of 95% confidence interval for $p$:

$p_H$ such that

$$\Pr(X \leq 0 \mid p = p_H) = 0.025$$

$\implies \Pr(X = 0 \mid p = p_H) = 0.025$

$\implies (1 - p_H)^n = 0.025$

$\implies 1 - p_H = \sqrt[n]{0.025}$

$\implies p_H = 1 - \sqrt[n]{0.025}$

In the case $n = 10$ and $X = 0$, the 95% CI for $p$ is $(0, 0.31)$. 
New York Times, Feb 3, 2004:

The department [of Agriculture] has not changed last year’s plans to test 40,000 cows nationwide this year, out of 30 million slaughtered. Janet Riley, a spokeswoman for the American Meat Institute, which represents slaughterhouses, called that “plenty sufficient from a statistical standpoint.”

Suppose that the 40,000 cows tested are chosen at random from the population of 30 million cows, and suppose that 0 (or 1, or 2) are found to be infected.

How many of the 30 million total cows would we estimate to be infected?

What is the 95% confidence interval for the total number of infected cows?

<table>
<thead>
<tr>
<th>No. infected</th>
<th>Obs’d</th>
<th>Est’d</th>
<th>95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0 – 2767</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>750</td>
<td>19 – 4178</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1500</td>
<td>182 – 5418</td>
</tr>
</tbody>
</table>
The case $X = n$

Suppose $X \sim \text{Binomial}(n, p)$ and we observe $X = n$.

Upper limit of 95% confidence interval for $p$: $\rightarrow 1$

Lower limit of 95% confidence interval for $p$:

$p_L$ such that

$$Pr(X \geq n \mid p = p_L) = 0.025$$

$\implies Pr(X = n \mid p = p_L) = 0.025$

$\implies (p_L)^n = 0.025$

$\implies p_L = \sqrt[n]{0.025}$

In the case $n = 25$ and $X = 25$, the 95% CI for $p$ is $(0.86, 1.00)$. 
Large n and medium p

Suppose $X \sim \text{Binomial}(n, p)$.

$E(X) = np$ \hspace{1cm} SD(X) = \sqrt{np(1-p)}$

$\hat{p} = \frac{X}{n}$ \hspace{1cm} $E(\hat{p}) = p$ \hspace{1cm} $SD(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$

For large $n$ and medium $p$, $\rightarrow \hat{p} \sim \text{Normal}\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$

Use 95% confidence interval $\hat{p} \pm 1.96 \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

$\rightarrow$ Unfortunately, this can behave poorly.

$\rightarrow$ Fortunately, you can just use \texttt{binom.test()}