Practice Problems (Sample Size and Power)

• Assume that we have \( n \) independent measurements from a Normal distribution with unknown mean \( \Delta \) and standard deviation 1, i.e. \( X_1, \ldots, X_n \sim N(\Delta, \sigma = 1) \).

1. What is the distribution of \( \bar{X} \)?
2. To test for \( H_0 : \Delta = 0 \) versus \( H_a : \Delta > 0 \), what is your test statistic, and what is its distribution if \( H_0 \) was true?
3. For which values of the test statistic do you reject the null hypothesis?
4. If we want to have 80\% power to detect a mean of \( \Delta = 0.5 \), how large must \( n \) be?

Solution:

1. \( \bar{X} \sim N(\Delta, 1/\sqrt{n}) \)
2. \( Z = \bar{X}/(1/\sqrt{n}) = \sqrt{n} \times \bar{X} \sim N(0, 1) \) under \( H_0 \). Note that when \( \Delta > 0 \), \( Z \sim N(\sqrt{n} \times \Delta, 1) \).
3. Since we have a one-sided test with the alternative \( H_a : \Delta > 0 \), we reject the null hypothesis only for large positive values. If we want to control the type I error rate at 5\%, we use 1.64 (\( \text{qnorm}(0.95) \)) as cut-off. So we reject the null hypothesis if our test statistic is larger than 1.64.
4. The test statistic \( Z \) is \( \sqrt{n} \times \bar{X} \), and its distribution is \( N(\sqrt{n} \times \Delta, 1) \). We reject the null hypothesis if \( Z > 1.64 \). Therefore, the power is

\[
Pr(Z > 1.64) = Pr\left\{ (\bar{X} - \sqrt{n} \times \Delta) > 1.64 - \sqrt{n} \times \Delta \right\} = Pr\left\{ \tilde{Z} > 1.64 - \Delta \sqrt{n} \right\}
\]

with \( \tilde{Z} \sim N(0, 1) \) under \( H_a \). We want the power to be 80\%. When taking a draw from a \( N(0, 1) \), we find that the chance of getting a value larger than -0.84 is equal to 80\% (\( \text{qnorm}(0.2) \) or \( \text{qnorm}(0.8, \text{lower}=\text{FALSE}) \)). Therefore, to have 80\% power to detect a mean of \( \Delta = 0.5 \), it follows from \( Pr(Z > -0.84) = 0.80 \) that

\[
1.64 - 0.5\sqrt{n} = -0.84 \iff n = (2 \times (1.64 + 0.84))^2 = 24.6
\]

Hence we need at least 25 samples. If you type

```r
> power.t.test(delta=0.5,power=0.8,sd=1,alternative="one.sided", type="one.sample")
```

in R, the answer is 26.2, i.e. that 27 samples are needed. This is because R assumes that \( \sigma = 1 \) is the true but unknown standard deviation, and thus, has to be estimated from the data. In other words, it uses a t-distribution for the sample size calculations, instead of the normal distribution, and that's why the sample size needed is slightly larger. The price for not knowing the standard deviation is 2 extra samples!