• Suppose we randomly sample 100 mosquitoes at a study site, and find that 44 carry a parasite. Derive the maximum likelihood estimate for the proportion of infected mosquitoes in the population.

Solution:
The distribution function for a Binomial($n, p$) is $P(X = x) = \binom{n}{x}p^x(1 - p)^{n-x}$. In our case, $n = 100$, and we observe $x = 44$. The likelihood is a function of the unknown parameter $p$, given the data. Hence

$$L(p) = \left(\begin{array}{c} 100 \\ 44 \end{array}\right) \times p^{44} \times (1 - p)^{100-44}$$

The log likelihood is

$$l(p) = \log \left( \left(\begin{array}{c} 100 \\ 44 \end{array}\right) \times p^{44} \times (1 - p)^{56} \right) = \log \left(\begin{array}{c} 100 \\ 44 \end{array}\right) + 44 \log(p) + 56 \log(1 - p)$$

To find the maximum of the log-likelihood, we take the first derivative of $l(p)$ with respect to $p$:

$$l'(p) = 0 + 44 \frac{1}{p} + 56 \frac{1}{1 - p} (-1) = \frac{44}{p} - \frac{56}{1 - p}$$

Setting the first derivative equal to zero yields

$$0 = \frac{44}{\hat{p}} - \frac{56}{1 - \hat{p}} \iff 0 = 44(1 - \hat{p}) - 56\hat{p} \iff 0 = 44 - 100\hat{p} \iff \hat{p} = \frac{44}{100}$$

Therefore, the maximum likelihood estimate for the proportion of infected mosquitoes in the population is 0.44.

• Suppose we take 3 random (independent) draws from a Poisson distribution, and obtain the numbers 18, 16, 23. Derive the maximum likelihood estimate for the Poisson parameter $\lambda$.

Solution:
The distribution function for a Poisson($\lambda$) is $P(X = x) = \exp(-\lambda) \times \lambda^x/x!$. Since we have three independent draws, the probability of observing those data is

$$P(X_1 = 18 \text{ and } X_2 = 16 \text{ and } X_1 = 23) = P(X = 18) \times P(X = 16) \times P(X = 23)$$

$$= \exp(-\lambda) \times \lambda^{18}/18! \times \exp(-\lambda) \times \lambda^{16}/16! \times \exp(-\lambda) \times \lambda^{23}/23!$$

$$= \exp(-\lambda)^3 \times \lambda^{18+16+23}/(18! \times 16! \times 23!)$$

$$= \exp(-\lambda)^3 \times \lambda^{57}/(\text{some really large constant})$$

This is the likelihood function, and the only unknown quantity is $\lambda$. The log likelihood is

$$l(\lambda) = \log \{ \exp(-\lambda)^3 \times \lambda^{57}/(\text{some really large constant}) \}$$

$$= -3\lambda + 57 \log(\lambda) - \log(\text{some really large constant}).$$
To find the maximum of the log-likelihood, we take the first derivative of $l(\lambda)$ with respect to $\lambda$:

$$l'(\lambda) = -3 + 57/\lambda$$

Setting the first derivative equal to zero yields

$$0 = -3 + 57/\hat{\lambda} \iff \hat{\lambda} = \frac{57}{3} = 19.$$

Therefore, the maximum likelihood estimate for the Poisson parameter $\lambda$ is 19.