Confidence Intervals for Proportions

Example


Place tick on clay island surrounded by water, with two capillary tubes: one treated with deer-gland-substance; one untreated.

<table>
<thead>
<tr>
<th>Tick sex</th>
<th>Leg</th>
<th>Deer sex</th>
<th>treated</th>
<th>untreated</th>
</tr>
</thead>
<tbody>
<tr>
<td>male</td>
<td>fore</td>
<td>female</td>
<td>24</td>
<td>5</td>
</tr>
<tr>
<td>female</td>
<td>fore</td>
<td>female</td>
<td>18</td>
<td>5</td>
</tr>
<tr>
<td>male</td>
<td>fore</td>
<td>male</td>
<td>23</td>
<td>4</td>
</tr>
<tr>
<td>female</td>
<td>fore</td>
<td>male</td>
<td>20</td>
<td>4</td>
</tr>
<tr>
<td>male</td>
<td>hind</td>
<td>female</td>
<td>17</td>
<td>8</td>
</tr>
<tr>
<td>female</td>
<td>hind</td>
<td>female</td>
<td>25</td>
<td>3</td>
</tr>
<tr>
<td>male</td>
<td>hind</td>
<td>male</td>
<td>21</td>
<td>6</td>
</tr>
<tr>
<td>female</td>
<td>hind</td>
<td>male</td>
<td>25</td>
<td>2</td>
</tr>
</tbody>
</table>

→ Is the tick more likely to go to the treated tube?
Test for a proportion

Suppose $X \sim \text{Binomial}(n, p)$.

Test $H_0 : p = \frac{1}{2}$ vs $H_a : p \neq \frac{1}{2}$.

Reject $H_0$ if $X \geq H$ or $X \leq L$.

Choose $H$ and $L$ such that
\[ \Pr(X \geq H \mid p = \frac{1}{2}) \leq \alpha/2 \quad \text{and} \quad \Pr(X \leq L \mid p = \frac{1}{2}) \leq \alpha/2. \]

Thus $\Pr(\text{Reject } H_0 \mid H_0 \text{ is true}) \leq \alpha$.

$\rightarrow$ The difficulty: The Binomial distribution is hard to work with. Because of its discrete nature, you can’t get exactly your desired significance level ($\alpha$).

Rejection region

Consider $X \sim \text{Binomial}(n=29, p)$.

Test of $H_0 : p = \frac{1}{2}$ vs $H_a : p \neq \frac{1}{2}$ at significance level $\alpha = 0.05$.

Lower critical value:
\[ q\text{binom}(0.025, 29, 0.5) = 9 \]
\[ \Pr(X \leq 9) = p\text{binom}(9, 29, 0.5) = 0.031 \rightarrow L = 8 \]

Upper critical value:
\[ q\text{binom}(0.975, 29, 0.5) = 20 \]
\[ \Pr(X \geq 20) = 1 - p\text{binom}(19, 29, 0.5) = 0.031 \rightarrow H = 21 \]

Reject $H_0$ if $X \leq 8$ or $X \geq 21$. (For testing $H_0 : p = \frac{1}{2}$, $H = n - L$)
Consider $X \sim \text{Binomial}(n=29, p)$.

Test of $H_0 : p = \frac{1}{2}$ vs $H_a : p \neq \frac{1}{2}$ at significance level $\alpha = 0.05$.

Reject $H_0$ if $X \leq 8$ or $X \geq 21$.

Actual significance level:

$$\alpha = \Pr(X \leq 8 \text{ or } X \geq 21 \mid p = \frac{1}{2})$$

$$= \Pr(X \leq 8 \mid p = \frac{1}{2}) + [1 - \Pr(X \leq 20 \mid p = \frac{1}{2})]$$

$$= \text{pbinom}(8, 29, 0.5) + 1 - \text{pbinom}(20, 29, 0.5)$$

$$= 0.024$$

If we used instead “Reject $H_0$ if $X \leq 9$ or $X \geq 20$”, the significance level would be

$$\text{pbinom}(9, 29, 0.5) + 1 - \text{pbinom}(19, 29, 0.5) = 0.061$$
Example 1

Observe $X = 24$ (for $n = 29$).

Reject $H_0: p = \frac{1}{2}$ if $X \leq 8$ or $X \geq 21$.

Thus we reject $H_0$ and conclude that the ticks were more likely to go to the deer-gland-substance-treated tube.

$$P\text{-value} = 2 \times \Pr(X \geq 24 \mid p = \frac{1}{2})$$
$$= 2 \times (1 - \text{pbinom}(23, 29, 0.5))$$
$$= 0.0005.$$

→ Alternatively: \text{binom.test}(24, 29)

Example 2

Observe $X = 17$ (for $n = 25$); assume $X \sim \text{Binomial}(n=25, p)$.

Test of $H_0: p = \frac{1}{2}$ vs $H_a: p \neq \frac{1}{2}$ at significance level $\alpha = 0.05$.

Rejection rule: Reject $H_0$ if $X \leq 7$ or $X \geq 18$.

$$\text{qbinom}(0.025, 25, 0.5) = 8$$
$$\text{pbinom}(8, 25, 0.5) = 0.054$$
$$\text{pbinom}(7, 25, 0.5) = 0.022$$

Significance level:

$$\text{pbinom}(7, 25, 0.5) + 1 - \text{pbinom}(17, 25, 0.5) = 0.043$$

Since we observed $X = 17$, we fail to reject $H_0$.

$$P\text{-value} = 2 \times (1 - \text{pbinom}(16, 25, 0.5)) = 0.11$$
Confidence interval for a proportion

Suppose $X \sim \text{Binomial}(n=29, p)$ and we observe $X = 24$.

Consider the test of $H_0 : p = p_0$ vs $H_a : p \neq p_0$.

We reject $H_0$ if

$$\Pr(X \leq 24 \mid p = p_0) \leq \alpha/2 \quad \text{or} \quad \Pr(X \geq 24 \mid p = p_0) \leq \alpha/2$$

95% confidence interval for $p$:

\[ \text{The set of } p_0 \text{ for which a two-tailed test of } H_0 : p = p_0 \text{ would not be rejected, for the observed data, with } \alpha = 0.05. \]

\[ \text{The “plausible” values of } p. \]

Example 1

$X \sim \text{Binomial}(n=29, p)$; observe $X = 24$.

Lower bound of 95% confidence interval:

Largest $p_0$ such that $\Pr(X \geq 24 \mid p = p_0) \leq 0.025$

Upper bound of 95% confidence interval:

Smallest $p_0$ such that $\Pr(X \leq 24 \mid p = p_0) \leq 0.025$

\[ \text{binom.test}(24, 29) \]

95% CI for $p$: $(0.642, 0.942)$

Note: $\hat{p} = 24/29 = 0.83$ is not the midpoint of the CI.
**Example 1**

Binomial(n=29, p=0.64)

Binomial(n=29, p=0.94)

**Example 2**

\( X \sim \text{Binomial}(n=25, p) \); observe \( X = 17 \).

Lower bound of 95% confidence interval:

\( p_L \) such that 17 is the 97.5 percentile of \( \text{Binomial}(n=25, p_L) \)

Upper bound of 95% confidence interval:

\( p_H \) such that 17 is the 2.5 percentile of \( \text{Binomial}(n=25, p_H) \)

\[ \rightarrow \text{binom.test}(17,25) \]

95% CI for \( p \): (0.465, 0.851)

Again, \( \hat{p} = 17/25 = 0.68 \) is not the midpoint of the CI
Example 2

The case $X = 0$

Suppose $X \sim \text{Binomial}(n, p)$ and we observe $X = 0$.

Lower limit of 95% confidence interval for $p$: $\rightarrow 0$

Upper limit of 95% confidence interval for $p$:

$p_H$ such that

$$
\Pr(X \leq 0 \mid p = p_H) = 0.025
$$

$$
\implies \Pr(X = 0 \mid p = p_H) = 0.025
$$

$$
\implies (1 - p_H)^n = 0.025
$$

$$
\implies 1 - p_H = \sqrt[n]{0.025}
$$

$$
\implies p_H = 1 - \sqrt[n]{0.025}
$$

In the case $n = 10$ and $X = 0$, the 95% CI for $p$ is $(0, 0.31)$. 
A mad cow example

New York Times, Feb 3, 2004:

The department [of Agriculture] has not changed last year’s plans to test 40,000 cows nationwide this year, out of 30 million slaughtered. Janet Riley, a spokeswoman for the American Meat Institute, which represents slaughterhouses, called that “plenty sufficient from a statistical standpoint.”

Suppose that the 40,000 cows tested are chosen at random from the population of 30 million cows, and suppose that 0 (or 1, or 2) are found to be infected.

How many of the 30 million total cows would we estimate to be infected?

What is the 95% confidence interval for the total number of infected cows?

<table>
<thead>
<tr>
<th>No. infected</th>
<th>Obs’d</th>
<th>Est'd</th>
<th>95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0 – 2767</td>
</tr>
<tr>
<td>1</td>
<td>750</td>
<td>19</td>
<td>19 – 4178</td>
</tr>
<tr>
<td>2</td>
<td>1500</td>
<td>182</td>
<td>182 – 5418</td>
</tr>
</tbody>
</table>

The case $X = n$

Suppose $X \sim \text{Binomial}(n, p)$ and we observe $X = n$.

Upper limit of 95% confidence interval for $p$: $\rightarrow 1$

Lower limit of 95% confidence interval for $p$:

$p_L$ such that

\[
\Pr(X \geq n \mid p = p_L) = 0.025
\]

\[
\implies \Pr(X = n \mid p = p_L) = 0.025
\]

\[
\implies (p_L)^n = 0.025
\]

\[
\implies p_L = \sqrt[n]{0.025}
\]

In the case $n = 25$ and $X = 25$, the 95% CI for $p$ is $(0.86, 1.00)$. 
Suppose $X \sim \text{Binomial}(n, p)$.

$$E(X) = np \quad \text{SD}(X) = \sqrt{n p(1 - p)}$$

$$\hat{p} = \frac{X}{n} \quad E(\hat{p}) = p \quad \text{SD}(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$$

For large $n$ and medium $p$, $\rightarrow \hat{p} \sim \text{Normal}\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$

Use 95% confidence interval $\hat{p} \pm 1.96 \sqrt{\frac{p(1-p)}{n}}$

$\rightarrow$ Unfortunately, this can behave poorly.

$\rightarrow$ Fortunately, you can just use `binom.test()`